

BIRATIONAL EQUIVALENCE OF HIGGS MODULI

MRIDUL MEHTA

ABSTRACT. In this paper, we study triples of the form (E, θ, ϕ) over a compact Riemann Surface, where (E, θ) is a Higgs bundle and ϕ is a global holomorphic section of the Higgs bundle. Our main result is an description of a birational equivalence which relates geometrically the moduli space of Higgs bundles of rank r and degree d to the moduli space of Higgs bundles of rank $r - 1$ and degree d .

1. INTRODUCTION

In [2], Bradlow introduced and studied the moduli space of pairs of the form (E, ϕ) over a compact Kähler manifold X , where E is a holomorphic vector bundle of fixed rank r and degree d over X and ϕ is a holomorphic section of E . The construction of the moduli space of these pairs involves a choice of linearization, which results in the notion of parameter dependent stability for these objects. This parameter is a real number taking values in the closed interval $\left[\frac{d}{r}, \frac{d}{r-1}\right]$. Consequently, the construction yields not one but a family of moduli spaces \mathcal{B}_τ of ' τ -stable' pairs. It was shown that for all but finitely many values of τ in the specified interval, the spaces \mathcal{B}_τ are birational. Further, when X is a Riemann surface, the τ stability condition forces the spaces $\mathcal{B}_{\frac{d}{r}}$ and $\mathcal{B}_{\frac{d}{r-1}}$ to be closely related to the moduli spaces $\mathcal{M}(r, d)$ and $\mathcal{M}(r-1, d)$ of vector bundles of rank r , degree d and rank $r-1$, degree d , respectively. As a result, the birational equivalence of the moduli spaces \mathcal{B}_τ relates geometrically the moduli spaces $\mathcal{M}(r, d)$ and $\mathcal{M}(r-1, d)$ over X . This setup, which was used by Thaddeus in his famous paper ([19]) provides a natural method to study the moduli space of vector bundles over a Riemann surface inductively using rank.

This paper provides a similar setup to study the moduli space of Higgs bundles. We consider objects of the form (E, θ, ϕ) over a Riemann surface X , where E is a holomorphic vector bundle of rank r and degree d over X , θ is a Higgs field and ϕ is a holomorphic global section of E . Constructing the moduli space for these objects leads to a notion of parameter dependent stability, where the parameter τ is a real number which takes values in the interval $\left[\frac{d}{r}, \frac{d}{r-1}\right]$. We show that for all but finitely many values of τ , these moduli spaces are birationally equivalent. Finally, we relate the moduli spaces of these triples at the end points of the interval to the moduli spaces $\mathcal{H}(r, d)$ and $\mathcal{H}(r-1, d)$ of $GL_r(\mathbb{C})$ -Higgs bundles over X , which completes the setup. The main results of this paper are the following.

Theorem 1.1. *For all noncritical values of τ in $\left(\frac{d}{r}, \frac{d}{r-1}\right)$, the spaces \mathcal{B}_τ of τ -stable Higgs triples are all birational.*

Let $\mathcal{H}_0(r, d) = \left\{ (\bar{\partial}_E, \theta) \in \mathcal{H}(r, d) \mid \det(\theta) = 0 \right\}$. We then have:

Theorem 1.2. *There exists a fibre space \mathcal{F} over $\mathcal{H}_0(r, d)$ which is birational to a projective bundle \mathcal{E} over $\mathcal{H}(r-1, d)$.*

Finally, it should be mentioned that much of this work has been motivated by the paper on stable pairs by Bradlow, Daskalopoulos and Wentworth [6].

2. PRELIMINARIES

We begin by fixing our notation, which we shall use for the rest of this paper. Let X be a compact Riemann surface of genus g . We fix a Kähler form ω on X , normalized so that $\int_X \omega = \text{Vol}(X) = 4\pi$. Let E be a fixed smooth (C^∞) complex vector bundle on X of rank r and degree d . We will assume that r and d are coprime. We shall denote by \mathcal{A} the affine space of all $\bar{\partial}_E$ operators on E .

In order to be rigorous, we use a similar convention as Atiyah and Bott ([1]), and for any Hermitian bundle V over X , by $\Omega^{p,q}(V)$ we will always mean the Banach space of sections of Sobolev class L^2_{k-p-q} , of the bundle of differential forms of type (p, q) with values in V (any $k \geq 2$ suffices). Then \mathcal{A} is the space of holomorphic structures on E differing from a fixed C^∞ one by an element of the Sobolev space $\Omega^{0,1}(\text{End } E)$. We shall denote by $E^{\bar{\partial}_E}$ the holomorphic bundle determined by any $\bar{\partial}_E \in \mathcal{A}$. Further, any such $\bar{\partial}_E$ also determines canonically a holomorphic structure on the bundle E^* , and hence a holomorphic structure on the bundle $\text{End } E \otimes \Omega^1_X$. By abuse of notation, we will denote all of these by $\bar{\partial}_E$. Given any Hermitian metric H on E , we shall denote by $F_{\bar{\partial}_E, H}$ the curvature of the metric connection on $E^{\bar{\partial}_E}$.

We shall denote by $\mathcal{G}^\mathbb{C}$ the complex gauge group of all complex automorphisms of E . So $\mathcal{G}^\mathbb{C}$ is the complexification of the unitary gauge group \mathcal{G} of automorphisms that preserve a fixed Hermitian metric on E . The space \mathcal{A} has a natural action of $\mathcal{G}^\mathbb{C}$ on it given by:

$$g \cdot \bar{\partial}_E = g \circ \bar{\partial}_E \circ g^{-1}$$

for any $g \in \mathcal{G}^\mathbb{C}$.

Recall that a $GL_r(\mathbb{C})$ -Higgs bundle of rank r and degree d on X is a pair of objects $(\bar{\partial}_E, \theta)$ where $\bar{\partial}_E \in \mathcal{A}$ as before is a holomorphic structure on E , while $\theta : E \rightarrow E \otimes \Omega^1_X$ is a holomorphic map (so $\theta \in H^0(\text{End } E^{\bar{\partial}_E} \otimes \Omega^1_X)$) such that $\theta \wedge \theta = 0$. In our case, since X is a Riemann surface, the condition $\theta \wedge \theta = 0$ is vacuous. Since the focus of our attention will only be $GL_r(\mathbb{C})$ -Higgs bundles, we will drop the prefix and refer to these simply as Higgs bundles. Given a Higgs bundle $(\bar{\partial}_E, \theta)$, we say that a holomorphic subbundle $F \subset E^{\bar{\partial}_E}$ is θ -invariant if $\theta(F) \subseteq F \otimes \Omega^1_X$. The Higgs bundle $(\bar{\partial}_E, \theta)$ is said to be semistable if

$$\mu(E') \leq \mu(E^{\bar{\partial}_E}) \quad \text{for all } \theta\text{-invariant proper holomorphic subbundles } E' \subset E^{\bar{\partial}_E},$$

and it is said to be stable if the above inequality is strict. As in the case of vector bundles, the complex gauge group $\mathcal{G}^\mathbb{C}$ acts naturally on the space of Higgs bundles over X :

$$g \cdot (\bar{\partial}_E, \theta) = (g \circ \bar{\partial}_E \circ g^{-1}, g \circ \theta \circ g^{-1})$$

for any $g \in \mathcal{G}^\mathbb{C}$. This action preserves the subset of stable Higgs bundles, and the quotient of this subset by the gauge group is the moduli space $\mathcal{H}(r, d)$ of Higgs bundles of rank r and degree d over X .

A stable pair of rank r and degree d on X is a pair of objects $(\bar{\partial}_E, \phi)$ where $\bar{\partial}_E \in \mathcal{A}$ as before is a holomorphic structure on E , while ϕ is a holomorphic section of E (so $\phi \in H^0(E^{\bar{\partial}_E})$). These were first studied by Bradlow in [2] and [3]. Given any real number τ , a pair $(\bar{\partial}_E, \phi)$ is said to be τ -semistable if

$$\begin{aligned} \mu(E') &\leq \tau \quad \text{for all holomorphic subbundles } E' \subseteq E^{\bar{\partial}_E}, \text{ and} \\ \mu(E^{\bar{\partial}_E}/E') &\geq \tau \quad \text{for all proper holomorphic subbundles } E' \subset E^{\bar{\partial}_E} \text{ that contain } \phi \\ &\text{i.e., } \phi \in H^0(E'). \end{aligned}$$

As usual, the pair is said to be τ -stable if the above inequalities are strict. It can be shown that the set of τ -semistable pairs is non-empty precisely when $\tau \in \left[\frac{d}{r}, \frac{d}{r-1}\right]$. For any fixed $\tau \in \left[\frac{d}{r}, \frac{d}{r-1}\right]$, the space of τ -stable pairs over X also has a natural action of the complex gauge group on it given by

$$g \cdot (\bar{\partial}_E, \phi) = (g \circ \bar{\partial}_E \circ g^{-1}, g \circ \phi)$$

for any $g \in \mathcal{G}^{\mathbb{C}}$, and the resulting quotient is the moduli space of τ -stable pairs over X .

Higgs bundles and stable pairs are both examples of augmented bundles that satisfy the Hitchin-Kobayashi correspondence. In the case of Higgs bundles (proved by Hitchin) this states that if a Higgs bundle $(\bar{\partial}_E, \theta)$ is stable, then the equation

$$F_{\bar{\partial}_E, H} + [\theta, \theta^{*H}] = \frac{d}{2r} i\omega \mathbf{I}$$

considered as an equation of the Hermitian metric H on E has a unique (up to scalar multiplication) smooth solution. Here θ^{*H} is the adjoint of θ with respect to the metric H defined by

$$(\theta u, v)_H = (u, \theta^{*H} v)_H,$$

and $[\cdot, \cdot]$ is the Lie bracket. Conversely, the existence of such a solution implies the polystability of the Higgs bundle. In the case of stable pairs, we have that if a pair $(\bar{\partial}_E, \phi)$ is τ -stable, then

$$i\Lambda F_{\bar{\partial}_E, H} + \frac{1}{2}(\phi \otimes \phi^{*H}) = \frac{\tau}{2} \mathbf{I}$$

considered as an equation of the Hermitian metric H on E has a unique (up to scalar multiplication) smooth solution. Here Λ is the adjoint (with respect to the metric H) of L (the Lefschetz operator given by the Kähler form ω on X). Again, there exists a suitable converse of this result.

The Hitchin-Kobayashi correspondence has been generalized adequately for our purposes by Bradlow, Garcia-Prada, and Riera in [7]. It is this interplay between stability and Hermitian metrics that allows us to interpret the various moduli spaces in more than one way.

3. HIGGS TRIPLES

We now introduce a new augmented bundle in gauge theory, which we shall use to study the moduli space of Higgs bundles. As before, given a Riemann surface X and the complex vector bundle E on X , we make the following definition.

Definition 3.1. A Higgs triple of rank r and degree d on X is a 3-tuple of objects $(\bar{\partial}_E, \theta, \phi)$ where $\bar{\partial}_E \in \mathcal{A}$ is a holomorphic structure on E , $\theta : E \rightarrow E \otimes \Omega_X^1$ is a holomorphic map (so θ is just a Higgs field), and ϕ is a holomorphic global section of $E^{\bar{\partial}_E}$ such that $\theta(\phi) = 0$.

Alternately, a Higgs triple may be thought of as a pair of objects (D'', ϕ) , where $D'' = \bar{\partial}_E + \theta$ is a Higgs bundle, and ϕ is a ‘holomorphic section of the Higgs bundle’ i.e., $D''(\phi) = 0$. This viewpoint makes the analogy between stable pairs and these triples more obvious in the way they relate to vector bundles and Higgs bundles respectively.

In order to construct a moduli space, we define the space of \mathcal{T} of Higgs triples:

$$\mathcal{T} = \left\{ (\bar{\partial}_E, \theta, \phi) \in \mathcal{A} \times \Omega^0(\text{End } E \otimes \Omega_X^1) \times \Omega^0(E) \mid \bar{\partial}_E(\theta) = 0, \bar{\partial}_E(\phi) = 0 \text{ and } \theta(\phi) = 0 \right\}.$$

The condition $\theta(\phi) = 0$ implies that the line subbundle $[\phi] \subset E^{\bar{\partial}_E}$ generated by ϕ is θ -invariant.

The space \mathcal{T} admits a natural action of the complex gauge group $\mathcal{G}^\mathbb{C}$. This allows us to study these triples up to isomorphism. The action is given by

$$g \cdot (\bar{\partial}_E, \theta, \phi) = (g \circ \bar{\partial}_E \circ g^{-1}, g \circ \theta \circ g^{-1}, g \circ \phi)$$

for any $g \in \mathcal{G}^\mathbb{C}$.

Next, we define the notion of stability for these objects. Although this notion is understood more easily using a real parameter τ , it may be defined intrinsically. In order to do this, we need the following. Let

$$\begin{aligned} \mu_M(\bar{\partial}_E, \theta) &= \sup \left\{ \mu(E') \mid E' \subseteq E^{\bar{\partial}_E} \text{ is a } \theta\text{-invariant holomorphic subbundle} \right\}, \\ \mu_m(\bar{\partial}_E, \theta, \phi) &= \inf \left\{ \mu(E/E') \mid E' \subset E^{\bar{\partial}_E} \text{ is a } \theta\text{-invariant proper} \right. \\ &\quad \left. \text{holomorphic subbundle containing } \phi \text{ i.e., } \phi \in \Omega^0(E') \right\}. \end{aligned}$$

Now we can state the definition of stability of these triples.

Definition 3.2. A Higgs triple $(\bar{\partial}_E, \theta, \phi) \in \mathcal{T}$ is said to be semistable if

$$\mu_M(\bar{\partial}_E, \theta) \leq \mu_m(\bar{\partial}_E, \theta, \phi).$$

The triple is said to be stable if the above inequality is strict. Moreover, if τ is any real number such that

$$\mu_M(\bar{\partial}_E, \theta) \leq \tau \leq \mu_m(\bar{\partial}_E, \theta, \phi)$$

then the triple $(\bar{\partial}_E, \theta, \phi)$ will be said to be τ -semistable. As before, the triple is said to be τ -stable if the inequalities are strict.

Clearly, a triple is semistable (resp. stable), if and only if there exists a τ such that it is τ -semistable (resp. τ -stable). Note that this notion of stability reduces to the notion of stability for stable pairs if the Higgs field vanishes ($\theta = 0$), and to the notion of stability for ordinary vector bundles if both the Higgs field and the global section vanish ($\theta = 0$ and $\phi = 0$).

Higgs triples, like other augmented bundles in gauge theory, also satisfy the Hitchin-Kobayashi correspondence. Suppose H is a Hermitian metric on E . Let

$(\bar{\partial}_E, \theta, \phi) \in \mathcal{T}$ be any triple. We consider the equation

$$(3.1) \quad i\Lambda (F_{\bar{\partial}_E, H} + [\theta, \theta^{*H}]) + \frac{1}{2}(\phi \otimes \phi^{*H}) = \frac{\tau}{2}\mathbf{I}.$$

Here ϕ^{*H} is the adjoint of ϕ , computed with respect to the metric H , and Λ is the adjoint (with respect to the metric H) of L (the Lefschetz operator given by ω on X), so that $\Lambda F_{D'', H}$ is an element of $\Omega^0(X, \text{End } E)$. τ is a real number, \mathbf{I} is the identity section of $\text{End}(E)$, and all the terms of the equation take values in $\Omega^0(\text{End } E)$. Following the terminology from Bradlow [2], we will call this the τ -vortex equation.

Note that in the absence of the Higgs field, the equation reduces to the vortex equation of Bradlow, while in the absence of the global section, the equation reduces to Hermitian-Yang-Mills equation of Hitchin, Simpson et al. As in these cases, the equation helps us determine certain preferred metrics on E .

Theorem 3.3. *Given E and X as above, let $(\bar{\partial}_E, \theta, \phi) \in \mathcal{T}$ be any Higgs triple. Suppose that there exists a real number τ such that $(\bar{\partial}_E, \theta, \phi)$ is a τ -stable Higgs triple (as defined above). Then the τ -vortex equation*

$$i\Lambda (F_{\bar{\partial}_E, H} + [\theta, \theta^{*H}]) + \frac{1}{2}(\phi \otimes \phi^{*H}) = \frac{\tau}{2}\mathbf{I}$$

considered as an equation for the Hermitian metric H on E has a unique smooth solution.

Conversely, suppose that for a given $\tau \in \mathbb{R}$, $\tau > 0$, there exists a Hermitian metric H on E such that the τ -vortex equation is satisfied by $(\bar{\partial}_E, \theta, \phi)$. Then $E^{\bar{\partial}_E}$ splits holomorphically as $E^{\bar{\partial}_E} = E_\phi \oplus E_s$ where

- (a) E_ϕ is θ -invariant, and contains the section ϕ (it is understood that E_ϕ obtains its holomorphic structure from $E^{\bar{\partial}_E}$)
- (b) $(E_\phi, \theta|_{E_\phi}, \phi)$ is a stable Higgs triple with $\mu_M(E_\phi, \theta|_{E_\phi}) < \tau < \mu_m(E_\phi, \theta|_{E_\phi}, \phi)$
- (c) E_s , if non-empty, is a direct sum of θ -invariant subbundles E_i
- (d) all the Higgs bundles (E_i, θ_i) are stable (in the usual sense of stability for Higgs bundles) with $\mu(E_i) = \tau$, where $\theta_i = \theta|_{E_i}$

We point out that a split $E^{\bar{\partial}_E} = E_\phi \oplus E_s$ cannot occur unless $\mu(E_s) = \tau$ is a rational number with denominator less than $\text{rk}(E)$. Hence, for *generic* values of τ , $E_s = \emptyset$.

Although the above theorem can be proved using methods similar to those of Simpson [15] and Bradlow [3] (done independently by the author, unpublished), it also follows as a special case from the recent work of Bradlow, Garcia-Prada and Riera [7]. The essential observation is that the moment map for the action of the unitary gauge group \mathcal{G} on the Kähler space $\mathcal{A} \times \Omega^1(\text{End } E) \times \Omega^0(E)$ is

$$(F_{\bar{\partial}_E, H} + [\theta, \theta^{*H}]) - \frac{i}{2}(\phi \otimes \phi^{*H})\omega,$$

where H is a fixed Hermitian metric on E .

The values of the parameter τ for which the solution set of the vortex equation is non-empty lie in a closed interval as in [6].

Proposition 3.4. *There is a solution to the τ -vortex equation only if $\frac{d}{r} \leq \tau \leq \frac{d}{r-1}$.*

Proof. By taking the trace of the τ -vortex equation and integrating over the Riemann surface X , we get

$$d + \frac{1}{2} \|\phi\|^2 = r\tau,$$

which implies that there is no solution unless

$$\frac{d}{r} \leq \tau.$$

For an upper bound, consider the θ -invariant line subbundle $[\phi] \subset E^{\bar{\partial}_E}$. If the vortex equation is satisfied, then either $(\bar{\partial}_E, \theta, \phi)$ is τ -stable, or it splits holomorphically into a direct sum of τ -stable pieces with prescribed slope. In the latter case, the τ -vortex equation is then satisfied by each piece separately so that either way $(\bar{\partial}_E, \theta, \phi)$ must be τ -semistable. Thus we have

$$\tau \leq \frac{\deg(E/[\phi])}{\text{rk}(E/[\phi])}$$

and since $\deg([\phi]) \geq 0$, we conclude that

$$\tau \leq \frac{d}{r-1}.$$

□

The converse to the above proposition is also true, and will be proved later.

4. MODULI SPACES

From the previous section we see that for generic values of τ , the set

$$\mathcal{V}_\tau = \left\{ (\bar{\partial}_E, \theta, \phi) \in \mathcal{T} \mid \Lambda F_{D'', H} - \frac{i}{2}(\phi \otimes \phi^{*H}) = -i\frac{\tau}{2}\mathbf{I} \text{ for some metric } H \right\}$$

consists exactly of the τ -stable Higgs triples. Further, upon fixing a Hermitian metric H on E , the unitary gauge group \mathcal{G} acts symplectically on \mathcal{T} , and the moment map for this action is given by

$$\Psi(\bar{\partial}_E, \theta, \phi) = \Lambda F_{D'', H} - \frac{i}{2}(\phi \otimes \phi^{*H}).$$

Hence for generic τ , the spaces \mathcal{V}_τ consist precisely of all the $\mathcal{G}^\mathbb{C}$ orbits through the triples in $\Psi^{-1}(-i\frac{\tau}{2}\mathbf{I})$. As a result, we may define the moduli space of τ -stable Higgs triples (for generic τ) in two different ways:

$$\mathcal{B}_\tau = \mathcal{V}_\tau / \mathcal{G}^\mathbb{C} = \Psi^{-1}(-i\frac{\tau}{2}\mathbf{I}) / \mathcal{G}.$$

Along the lines of Bradlow et al. [6], we construct a ‘master space’ which will contain stable and semi-stable Higgs triples for all values of τ . To do this, as in [6], we use a moment map to construct $\hat{\mathcal{B}}$: we replace the full unitary gauge group \mathcal{G} by a subgroup \mathcal{G}_0 which has a $U(1)$ quotient and whose Lie algebra is the L^2 orthocomplement of the constant multiples of the identity. Denoting the new moment map by Ψ_0 , we then obtain $\hat{\mathcal{B}}$ as $\Psi_0^{-1}(0)/\mathcal{G}_0$ symplectically. Finally, we obtain the complex structure on $\hat{\mathcal{B}}$ by looking at $\mathcal{G}_0^\mathbb{C}$ orbits through the triples in $\Psi_0^{-1}(0)$.

For details on the construction of the subgroups \mathcal{G}_0 and $\mathcal{G}_0^{\mathbb{C}}$, we refer the reader to section 2.2 in [6].

The complex structure of $\hat{\mathcal{B}}$ is essentially obtained from considering the action of $\mathcal{G}_0^{\mathbb{C}}$ on an appropriate subset of \mathcal{T} . Consequently we first focus our attention on the action of $\mathcal{G}^{\mathbb{C}}$ on this subset, a slight modification of which will lead us to our goal. First, we define

$$\mathcal{T}^* = \left\{ (\bar{\partial}_E, \theta, \phi) \in \mathcal{T} \mid (\bar{\partial}_E, \theta) \text{ is semistable if } \phi = 0 \right\}.$$

Then \mathcal{T}^* is an open subset of \mathcal{T} . Next, for any $\bar{\partial}'_E \in \mathcal{A}$, let

$$\mathcal{T}_{\bar{\partial}'_E}^* = \{ (\bar{\partial}_E, \theta, \phi) \in \mathcal{T}^* \mid \bar{\partial}_E = \bar{\partial}'_E \}.$$

We consider the infinitesimal deformations of pairs (θ, ϕ) inside the space $\mathcal{T}_{\bar{\partial}_E}^*$ for a fixed $\bar{\partial}_E \in \mathcal{A}$. In order to do this, we look at the complex

$$(\text{End } E \otimes K) \oplus E \longrightarrow E \otimes K,$$

where the map is given by

$$(v, w) \mapsto v\phi + \theta w.$$

Let $\mathbf{N}^{(\bar{\partial}_E, \theta, \phi)}$ be the resulting long exact sequence. We define

$$\mathcal{T}_{\bar{\partial}_E}^{**} = \left\{ (\bar{\partial}_E, \theta, \phi) \in \mathcal{T} \mid H^2(\mathbf{N}^{(\bar{\partial}_E, \theta, \phi)}) = 0 \right\}.$$

Then $\mathcal{T}_{\bar{\partial}_E}^{**}$ is an open subset of $\mathcal{T}_{\bar{\partial}_E}^*$. Now let

$$\mathcal{T}^{**} = \bigcup_{\bar{\partial}_E \in \mathcal{A}} \mathcal{T}_{\bar{\partial}_E}^{**}.$$

This space \mathcal{T}^{**} is an open subset of \mathcal{T}^* . For purely technical reasons (which will become clear later), we shall restrict our attention to this subset of \mathcal{T} .

Now we use the standard approach of infinitesimal deformations to compute the obstruction to $\mathcal{T}^{**}/\mathcal{G}^{\mathbb{C}}$ being a manifold. These arise from the hypercohomology of the complex $\mathbf{End}^{(\bar{\partial}_E, \theta, \phi)}$:

$$\text{End } E \longrightarrow (\text{End } E \otimes K) \oplus E \longrightarrow E \otimes K,$$

where the maps are given by

$$u \mapsto ([u, \theta], u\phi) \quad \text{and} \quad (v, w) \mapsto v\phi + \theta w,$$

respectively. The complex which allows us to compute hypercohomology then becomes

$$\begin{aligned} 0 \longrightarrow \Omega^0(\text{End } E) &\xrightarrow{d_1} \Omega^{0,1}(\text{End } E) \oplus \Omega^{1,0}(\text{End } E) \oplus \Omega^0(E) \\ &\xrightarrow{d_2} \Omega^{1,1}(\text{End } E) \oplus \Omega^{0,1}(E) \oplus \Omega^{1,0}(E) \xrightarrow{d_3} \Omega^{1,1}(E) \longrightarrow 0, \end{aligned}$$

where the maps d_1 , d_2 and d_3 are given by

$$\begin{aligned} d_1(u) &= (-\bar{\partial}_E(u), [u, \theta], u\phi) \\ d_2(\alpha, \beta, \gamma) &= ([\alpha, \theta] + \bar{\partial}_E(\beta), \alpha\phi + \bar{\partial}_E(\gamma), \beta\phi + \theta\gamma) \\ d_3(\lambda, \rho, \sigma) &= \lambda\phi + \theta\rho + \bar{\partial}_E(\sigma). \end{aligned}$$

We shall call the above complex $\mathbf{C}^{(\bar{\partial}_E, \theta, \phi)}$.

Proposition 4.5. *The cup product from $H^1(\mathbf{C}^{(\bar{\partial}_E, \theta, \phi)}) \times H^1(\mathbf{C}^{(\bar{\partial}_E, \theta, \phi)}) \rightarrow H^2(\mathbf{C}^{(\bar{\partial}_E, \theta, \phi)})$ vanishes.*

Proof. This follows now from smoothness of the Higgs moduli space and the moduli space of stable pairs, and our construction of \mathcal{T}^{**} . \square

Now we look at the action of $\mathcal{G}_0^{\mathbb{C}}$ on \mathcal{T}^{**} . This requires us to restrict to the following subcomplex of $\mathbf{C}^{(\bar{\partial}_E, \theta, \phi)}$:

$$\begin{aligned} 0 \longrightarrow \Omega^0(\text{End } E)_0 &\xrightarrow{d_1} \Omega^{0,1}(\text{End } E) \oplus \Omega^{1,0}(\text{End } E) \oplus \Omega^0(E) \\ &\xrightarrow{d_2} \Omega^{1,1}(\text{End } E) \oplus \Omega^{0,1}(E) \oplus \Omega^{1,0}(E) \xrightarrow{d_3} \Omega^{1,1}(E) \longrightarrow 0, \end{aligned}$$

which we shall denote by $\mathbf{C}_0^{(\bar{\partial}_E, \theta, \phi)}$. The only difference is in the first term which is now $\Omega^0(\text{End } E)_0$. Recall that $\Omega^0(\text{End } E)_0$ was the L^2 -orthogonal complement of the constant multiples of the identity in $\Omega^0(\text{End } E)$. The important point to note here is that the cup product from $H^1(\mathbf{C}_0^{(\bar{\partial}_E, \theta, \phi)}) \times H^1(\mathbf{C}_0^{(\bar{\partial}_E, \theta, \phi)}) \rightarrow H^2(\mathbf{C}_0^{(\bar{\partial}_E, \theta, \phi)})$ still vanishes. Consequently we have the following.

Proposition 4.6. *\mathcal{T}^{**} is a smooth submanifold of $\mathcal{A} \times \Omega^{0,1}(E) \times \Omega^0(E)$.*

Definition 4.7. *We define a triple $(\bar{\partial}_E, \theta, \phi) \in \mathcal{T}^{**}$ to be simple if $H^0(C_0^{(\bar{\partial}_E, \theta, \phi)}) = 0$. Denote by \mathcal{T}_σ the set of all simple triples in \mathcal{T}^{**} .*

Note that \mathcal{T}_σ is open in \mathcal{T}^{**} . This allows us to conclude the following.

Proposition 4.8. *$\mathcal{T}_\sigma/\mathcal{G}_0^{\mathbb{C}}$ is a complex manifold. Moreover, we have the identification*

$$T_{[\bar{\partial}_E, \theta, \phi]}(\mathcal{T}_\sigma/\mathcal{G}_0^{\mathbb{C}}) = H^1(\mathbf{C}_0^{(\bar{\partial}_E, \theta, \phi)}).$$

We now study the symplectic structure on $\mathcal{T}_\sigma/\mathcal{G}_0^{\mathbb{C}}$.

Proposition 4.9. *The moment map for the action of the subgroup \mathcal{G}_0 on \mathcal{T}^{**} is given by*

$$\Psi_0(\bar{\partial}_E, \theta, \phi) = \pi^\perp \Psi(\bar{\partial}_E, \theta, \phi) = \Psi(\bar{\partial}_E, \theta, \phi) - \frac{1}{r} \int_X \text{Tr } \Psi(\bar{\partial}_E, \theta, \phi) \cdot \mathbf{I}$$

Proof. We observe that $\Psi_0 = j^* \Psi$ where $j : \mathcal{G}_0 \rightarrow \mathcal{G}_0$ is the inclusion. For details, see section 2.4 in [6]. \square

We are now ready to define the master space $\hat{\mathcal{B}}$.

Definition 4.10. *Let*

$$\hat{\mathcal{B}} = (\Psi_0^{-1}(0) \cap \mathcal{T}^*)/\mathcal{G}_0$$

be the Marsden-Weinstein reduction by the symplectic action of \mathcal{G}_0 and

$$\hat{\mathcal{B}}_0 = (\Psi_0^{-1}(0) \cap \mathcal{T}_\sigma)/\mathcal{G}_0.$$

Proposition 4.11. *The space $\hat{\mathcal{B}}$ is a Hausdorff topological space. The space $\hat{\mathcal{B}}_0$ is a Hausdorff symplectic manifold.*

Proof. The first follows in an identical manner to the similar result in [6], while the second is a consequence of the Marsden-Weinstein reduction theorem for Banach spaces. \square

Finally, in order to obtain the complex structure on $\hat{\mathcal{B}}_0$, we define

Definition 4.12. Let $\mathcal{V}_0 \subset \mathcal{T}^{**}$ denote the subset of $\mathcal{G}_0^{\mathbb{C}}$ orbits through points in $\Psi_0^{-1}(0)$, that is

$$\mathcal{V}_0 = \left\{ (\bar{\partial}_E, \theta, \phi) \in \mathcal{T}^{**} \mid \Psi_0(g(\bar{\partial}_E, \theta, \phi)) = 0 \text{ for some } g \in \mathcal{G}_0^{\mathbb{C}} \right\}.$$

Note that $\mathcal{V}_0 \cap \mathcal{T}_\sigma$ is an open subset of \mathcal{T}_σ . Using the same technique as in [6], it also follows that $\mathcal{V}_0 \cap \mathcal{T}_\sigma$ is connected. Further, there is a bijective correspondence between $\hat{\mathcal{B}}_0$ and $(\mathcal{V}_0 \cap \mathcal{T}_\sigma)/\mathcal{G}_0^{\mathbb{C}}$, so that using propositions 4.8 and 4.11, we obtain the following.

Proposition 4.13. $\hat{\mathcal{B}}_0 = (\mathcal{V}_0 \cap \mathcal{T}_\sigma)/\mathcal{G}_0^{\mathbb{C}}$ is a smooth, Hausdorff manifold.

5. S^1 ACTION AND MORSE THEORY

A key feature of the master space $\hat{\mathcal{B}}$ is that it carries a natural S^1 -action. This arises from the quotient $\mathcal{G}/\mathcal{G}_0 = U(1)$. The action is given by:

$$e^{i\rho} \cdot [\bar{\partial}_E, \theta, \phi] = [\bar{\partial}_E, \theta, g_\rho \phi].$$

Here g_ρ denotes the gauge transformation $\text{diag}(e^{i\rho/r}, \dots, e^{i\rho/r})$. As shown in [6], this action is well-defined and independent of the choice of the r -th root of unity, for if $h = e^{2\pi i/r} \cdot \mathbf{I}$, then $h \in \mathcal{G}_0$ and

$$[\bar{\partial}_E, \theta, h\phi] = [h^{-1}\bar{\partial}_E, h^{-1}\theta, \phi] = [\bar{\partial}_E, \theta, \phi].$$

Proposition 5.1. The action of $U(1)$ on $\hat{\mathcal{B}}_0$ is holomorphic and symplectic. The moment map for the action is given by:

$$\hat{f}[\bar{\partial}_E, \theta, \phi] = -2\pi i \left(\frac{\|\phi\|^2}{4\pi r} + \mu(E) \right).$$

Further, this action extends continuously to $\hat{\mathcal{B}}$ as does the moment map \hat{f} .

Proof. The proof is identical to the one in [6]. The only observation which one needs is that the trace of the term contributed by the Higgs field is zero i.e.,

$$\text{Tr}([\theta, \theta^{*H}]) = 0.$$

□

For convenience, we define $f : \hat{\mathcal{B}} \rightarrow \mathbb{R}$ by

$$f = -\frac{1}{2\pi i} \hat{f}.$$

The moment map \hat{f} is the same as the one obtained in [6]. The next proposition summarizes the essential properties of f . The proofs are identical to proposition (2.14) in [6].

Proposition 5.2. (i) The image of f is the interval $\left[\frac{d}{r}, \frac{d}{r-1} \right]$.

(ii) The critical points of f on $\hat{\mathcal{B}}_0$ are exactly the fixed points of the $U(1)$ action. The critical values of f are precisely the image under f of the fixed point set of the $U(1)$ action on $\hat{\mathcal{B}}$.

(iii) If τ is a regular value of f , then the space $f^{-1}(\tau)/U(1)$ is \mathcal{B}_τ , the moduli space of τ -stable Higgs triples.

Proof. (i) τ is in the image of f if and only if the equation $\Psi((\bar{\partial}_E, \theta, \phi)) = -\frac{i\tau}{2}\mathbf{I}$ has a solution. By proposition 3.4, the range for τ is in $[d/r, d/(r-1)]$. Next, proposition 5.3 gives explicit elements of the spaces at the end points, $\mathcal{B}_{\frac{d}{r}}$ and $\mathcal{B}_{\frac{d}{r-1}}$. Therefore, since $\mathcal{V}_0 \cup \mathcal{T}_\sigma$ is connected, the result follows. (ii) This follows from the fact that \hat{f} is a moment map for the S^1 action. (iii) See [6]. \square

Next, we look at the level sets $f^{-1}(\tau)$ when τ is a critical value in $\left[\frac{d}{r}, \frac{d}{r-1}\right]$. We start with a simple but important observation.

Proposition 5.3. *The level set corresponding to the minimum is precisely the moduli space of semistable Higgs bundles of degree d and rank r :*

$$f^{-1}\left(\frac{d}{r}\right) = \mathcal{M}_{Higgs}(r, d),$$

while the level set corresponding to the maximum is precisely the moduli space of semistable Higgs bundles of degree d and rank $r-1$:

$$f^{-1}\left(\frac{d}{r-1}\right) = \mathcal{M}_{Higgs}(r-1, d).$$

Proof. By construction, $f^{-1}(d/r)$ consists exactly of (d/r) -semistable triples. However, the standard trick of taking the trace of the vortex equation and integrating over X shows that ϕ is forced to be identically 0 when $\tau = d/r$. Then (d/r) -semistable is the same as semistability in the usual sense of a Higgs bundle. To see this for the maximum level set, consider a solution $(\bar{\partial}_E, \theta, \phi)$ of the vortex equation with $\tau = d/(r-1)$. Then by theorem 3.3, $(\bar{\partial}_E, \theta, \phi)$ is either stable or splits holomorphically. If the triple is stable, then since the line subbundle $[\phi] \subset E^{\bar{\partial}_E}$ is θ -invariant, the stability criteria gives us

$$\mu(E/[\phi]) > d/(r-1) \Rightarrow \deg([\phi]) < 0,$$

which is a contradiction. Hence the triple (D'', ϕ) splits. But we also have

$$\mu(E/[\phi]) = d/(r-1) \Rightarrow \deg([\phi]) = 0.$$

Theorem 3.3 and its proof in the converse direction now forces ϕ to be a constant section of a trivial line subbundle. This means that $E^{\bar{\partial}_E}$ splits as $E^{\bar{\partial}_E} = \mathcal{O} \oplus E_s$, where $(E_s, \theta|_{E_s})$ is a semistable Higgs bundle of degree d and rank $r-1$. Further, as in [6], due to theorem 3.3 the Higgs bundle $(E_s, \theta|_{E_s})$ is a direct sum of stable Higgs bundles all of the same slope, so that it is equal to its graded S -equivalence class in $\mathcal{M}_{Higgs}(r-1, d)$. Thus the map $(\bar{\partial}_E, \theta, \phi) = (\mathcal{O} \oplus E_s \mapsto (E_s, \theta|_{E_s}))$ establishes the correspondence between $f^{-1}(d/(r-1))$ and $\mathcal{M}_{Higgs}(r-1, d)$. \square

We now turn our attention to the level sets corresponding to the critical values in the interior of the interval $\left[\frac{d}{r}, \frac{d}{r-1}\right]$. To this end we make the following definition.

Definition 5.4. *Let $Fix(\hat{\mathcal{B}})$ be the $U(1)$ fixed point set in $\hat{\mathcal{B}}$. Then for any critical value $\tau \in \left[\frac{d}{r}, \frac{d}{r-1}\right]$, define*

$$\mathcal{Z}_\tau = f^{-1}(\tau) \cap Fix(\hat{\mathcal{B}}).$$

Let $\tau = \frac{p}{q}$ be a critical value such that $\frac{d}{r} < \tau < \frac{d}{r-1}$. Then for any $(\bar{\partial}_E, \theta, \phi) \in \mathcal{Z}_\tau$, we have $\mu_M(\bar{\partial}_E, \theta) = \frac{p}{q} = \mu_m((\bar{\partial}_E, \theta, \phi))$. Further, theorem 3.3 allows us to make certain immediate observations which we summarize in the next proposition.

Proposition 5.5. *Let $\frac{p}{q} \in (\frac{d}{r}, \frac{d}{r-1})$ be a critical value, and $(\bar{\partial}_E, \theta, \phi) \in f^{-1}(\frac{p}{q})$. As in theorem 3.3, since $(\bar{\partial}_E, \theta, \phi)$ satisfies the vortex equation, we know that $E^{\bar{\partial}_E}$ splits holomorphically as $E^{\bar{\partial}_E} = E_\phi \oplus E_s$ where E_s is possibly a direct sum of bundles E_i . Let (r_ϕ, d_ϕ) and (r_i, d_i) be the rank and degree of E_ϕ and E_i , respectively. Suppose $(\bar{\partial}_E, \theta, \phi)$ is not $\frac{p}{q}$ -stable. Then the following holds:*

- (i) $\frac{d_i}{r_i} = \frac{d - d_\phi}{r - r_\phi} = \frac{p}{q}$
- (ii) $r_\phi + \sum_i r_i = r$
- (iii) $\frac{d_\phi}{r_\phi} < \frac{p}{q} < \frac{d_\phi}{r_\phi - 1}$

Conversely, given any stable Higgs triple $(E_\phi, \theta_\phi, \phi)$ and stable Higgs bundles (E_i, θ_i) so that the above conditions are satisfied, we have a representative for a fixed point $(\bar{\partial}_E, \theta, \phi)$ in $f^{-1}(\frac{p}{q})$ where $(E^{\bar{\partial}_E}, \theta) = (E_\phi, \theta_\phi) \oplus \bigoplus_i (E_i, \theta_i)$.

Proof. In the forward direction, since $(\bar{\partial}_E, \theta, \phi)$ is not $\frac{p}{q}$ -stable but just semistable, (i) and (ii) are obvious consequences of theorem 3.3. (iii) follows from applying the discussion on the possible range of the parameter τ to the $\frac{p}{q}$ -stable triple $(\bar{\partial}_{E_\phi}, \theta_\phi, \phi)$.

To see the converse, we use the same key theorem to find metrics on the various pieces E_ϕ and the bundles E_i so that the $\frac{p}{q}$ vortex equation holds for the various pieces. Then we piece these metrics together to obtain a metric H on $E^{\bar{\partial}_E} = E_\phi \oplus \bigoplus_i E_i$ so that the resulting triple $(\bar{\partial}_E, \theta, \phi)$ satisfies the $\frac{p}{q}$ -vortex equation. However, since this triple is only $\frac{p}{q}$ -semistable, it is in $f^{-1}(\frac{p}{q})$. \square

In order to investigate the level sets at the critical points, we stratify them in the following manner. Given any critical value $\tau = \frac{p}{q}$, let

$$\mathcal{I}_\tau = \left\{ (d_\phi, r_\phi, r_1, \dots, r_n) \in \mathbb{Z}^{n+2} \mid \frac{p}{q} = \tau = \frac{d - d_\phi}{r - r_\phi} \text{ and conditions (ii), (iii) of the previous proposition are satisfied} \right\}.$$

Note that given such an $(n+2)$ -tuple $(d_\phi, r_\phi, r_1, \dots, r_n)$, the degrees d_i are determined uniquely using condition (i) in the previous proposition.

This allows us to write \mathcal{Z}_τ as a disjoint union of sets $\mathcal{Z}(d_\phi, r_\phi, r_1, \dots, r_n)$ where

$$\mathcal{Z}(d_\phi, r_\phi, r_1, \dots, r_n) = \left\{ (\bar{\partial}_E, \theta, \phi) \in \mathcal{Z}_\tau \mid (E^{\bar{\partial}_E}, \theta) = (E_\phi, \theta_\phi) \oplus \bigoplus_i (E_i, \theta_i) \right\}.$$

Hence we get

$$\mathcal{Z}_\tau = \bigcup_{(d_\phi, r_\phi, r_1, \dots, r_n) \in \mathcal{I}_\tau} \mathcal{Z}(d_\phi, r_\phi, r_1, \dots, r_n).$$

By using the following convention we can include in the above discussion the critical values at the end points of the interval i.e., $\tau = \frac{d}{r}$ and $\tau = \frac{d}{r-1}$. When $\tau = \frac{d}{r}$, we will set $d_\phi = 0$ and $r_\phi = 0$, and when $\tau = \frac{d}{r-1}$, we will set $d_\phi = 0$ and $r_\phi = 1$.

6. ALGEBRAIC STRATIFICATION

In this section we define a natural stratification of $\hat{\mathcal{B}}$. We do this by defining two different filtrations on any Higgs triple in the master space, which arise as a consequence of the definition of stability for such objects. These filtrations are similar to the ones defined in [6], except we require that the relevant subbundles be invariant under the Higgs field.

Proposition 6.1. *(The μ_- -filtration). Let $(\bar{\partial}_E, \theta, \phi)$ be a stable Higgs triple. There is a filtration of E by θ -invariant subbundles*

$$(6.1) \quad 0 \subset E_\phi = F_0 \subset F_1 \subset \cdots \subset F_n = E$$

such that the following conditions hold:

- (a) $\phi \in H^0(E_\phi)$, $(E_\phi, \theta_\phi, \phi)$ is a stable Higgs triple (where $\theta_\phi = \theta|_{E_\phi}$), and $\mu_M(E_\phi, \theta_\phi) < \mu_m(E, \theta, \phi) < \mu_m(E_\phi, \theta_\phi, \phi)$,
- (b) for $1 \leq i \leq n$, the quotients $(F_i/F_{i-1}, \theta_i/\theta_{i-1})$ are stable Higgs bundles each of slope $\mu_m(E, \theta, \phi)$ (where $\theta_i = \theta|_{F_i}$),
- (c) E_ϕ has minimal rank such that (a) and (b) are satisfied.

Consequently, the subbundle E_ϕ is uniquely determined and the graded object

$$gr^-(\bar{\partial}_E, \theta) = (E_\phi, \theta_\phi) \oplus (F_1/F_0, \theta_1/\theta_0) \oplus \cdots \oplus (F_n/F_{n-1}, \theta_n/\theta_{n-1})$$

is unique up to isomorphism of $(F_1/F_0, \theta_1/\theta_0) \oplus \cdots \oplus (F_n/F_{n-1}, \theta_n/\theta_{n-1})$.

Before we can prove this, we need the following.

Lemma 6.2. *Let $(\bar{\partial}_E, \theta, \phi)$ be a stable triple. Let $E_\phi \subset E^{\bar{\partial}_E}$ be a θ -invariant subbundle such that $\phi \in H^0(E_\phi)$ and $\mu(E/E_\phi) = \mu_m(E, \theta, \phi)$. Let $\theta_\phi = \theta|_{E_\phi}$. Then*

- (a) $\mu_M(E_\phi, \theta_\phi) \leq \mu_M(E, \theta)$,
- (b) $\mu_m(E_\phi, \theta_\phi, \phi) \leq \mu_m(E, \theta, \phi)$ and the inequality is strict if E_ϕ has minimal rank among all subbundles satisfying the hypotheses of this Lemma,
- (c) the Higgs triple $(E_\phi, \theta_\phi, \phi)$ is stable,
- (d) $(E/E_\phi, \theta/\theta_\phi)$ is a semistable Higgs bundle,
- (e) $\mu(E_\phi) < \mu_m(E, \theta, \phi)$,
- (f) If E_ϕ has minimal rank among all the θ -invariant subbundles satisfying the hypotheses of this Lemma, and E'_ϕ is any other θ -invariant subbundle satisfying the same, then $E_\phi \subset E'_\phi$.

Proof. (a) This follows by definition of μ_M .

(b) The proof here is identical to [6] with a small modification: we consider only Higgs field invariant subbundles. This however, does not affect the argument.

(c) Using (a), (b) and the fact that (E, θ, ϕ) is stable, it follows that $(E_\phi, \theta_\phi, \phi)$ is stable.

(d) Suppose $(E/E_\phi, \theta/\theta_\phi)$ is not semistable as a Higgs bundle. Then there is some (θ/θ_ϕ) -inv. subbundle $F \subset E/E_\phi$ such that $\mu(F) = \mu_M(E/E_\phi, \theta/\theta_\phi) > \mu(E/E_\phi)$. Consider the Higgs extension of (E_ϕ, θ_ϕ) by (F, θ_F) where $\theta_F = (\theta/\theta_\phi)|_F$. This gives us an exact sequence of Higgs bundles

$$0 \longrightarrow (E_\phi, \theta_\phi) \longrightarrow (E', \theta') \longrightarrow (F, \theta_F) \longrightarrow 0.$$

Recall that an exact sequence of Higgs bundles as above is simply two exact sequences with maps between them so that the resulting diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_\phi & \longrightarrow & E' & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow \theta_\phi & & \downarrow \theta' & & \downarrow \theta_F & & \\ 0 & \longrightarrow & E_\phi \otimes K & \longrightarrow & E' \otimes K & \longrightarrow & F \otimes K & \longrightarrow & 0 \end{array}$$

where K is the canonical bundle on the Riemann surface X . Note that by construction E' is θ invariant, and we let $\theta' = \theta|_{E'}$. The rest of the argument works the same way as in [6].

- (e) Since (E, θ, ϕ) is stable, $\mu(E_\phi) \leq \mu_M(E, \theta) < \mu_m(E, \theta, \phi)$.
- (f) Suppose E_ϕ and E'_ϕ are as in statement (f) of the proposition. Let $\theta'_\phi = \theta|_{E'_\phi}$. We observe that the inclusion $(E_\phi, \theta_\phi) \rightarrow (E, \theta)$ and projection $(E, \theta) \rightarrow (E/E'_\phi, \theta/\theta'_\phi)$ are both morphisms of Higgs bundles. Hence, so is the composition $(E_\phi, \theta_\phi) \rightarrow (E/E'_\phi, \theta/\theta'_\phi)$. Taking the kernel and the image of this map we get the following exact sequence of Higgs bundles:

$$0 \longrightarrow (N, \theta_N) \longrightarrow (E_\phi, \theta_\phi) \longrightarrow (L, \theta_L) \longrightarrow 0,$$

where N and L are respectively the kernel and the image, $\theta_N = \theta_\phi|_N$ and $\theta_L = (\theta/\theta'_\phi)|_L$. The rest of the argument proceeds as in [6]. \square

Proof of Proposition 6.1. The previous Lemma allows us to find a unique θ -invariant subbundle $E_\phi \subset E$ of minimal rank such that

- (i) $\phi \in H^0(E_\phi)$,
- (ii) $\mu_M(E_\phi, \theta_\phi) < \mu_m(E, \theta, \phi) < \mu_m(E_\phi, \theta_\phi, \phi)$ (in particular $(E_\phi, \theta_\phi, \phi)$ is a stable Higgs triple),
- (iii) $\mu(E/E_\phi) = \mu_m(E, \theta, \phi)$,
- (iv) $(E/E_\phi, \theta/\theta_\phi)$ is a semistable Higgs bundle.

Now let

$$0 \subset (Q_1, \eta_1) \subset \cdots \subset (Q_n, \eta_n) = (E/E_\phi, \theta/\theta_\phi)$$

be the Harder-Narasimhan (HN) filtration for the Higgs bundle $(E/E_\phi, \theta/\theta_\phi)$. Note that the bundles Q_i are all θ/θ_ϕ -invariant and $\eta_i = (\theta/\theta_\phi)|_{Q_i}$. Let $\pi : (E, \theta) \rightarrow (E/E_\phi, \theta/\theta_\phi)$ be the projection map (which as discussed earlier is a morphism of Higgs bundles). Define $(F_i, \theta_i) = \pi^{-1}(Q_i, \eta_i)$. This gives us the filtration we seek. \square

This allows us to make the following definition.

Definition 6.3. For any stable Higgs triple $(\bar{\partial}_E, \theta, \phi)$, we define the μ_- grading to be given by

$$gr^-(\bar{\partial}_E, \theta, \phi) = (gr^-(\bar{\partial}_E, \theta), \phi),$$

where $gr^-(\bar{\partial}_E, \theta)$ is as above.

Proposition 6.4. (The μ_+ -filtration) Let $(\bar{\partial}_E, \theta, \phi)$ be a stable Higgs triple. There is a filtration of E by θ -invariant subbundles

$$(6.2) \quad 0 = F_0 \subset F_1 \subset \cdots \subset F_n \subset F_{n+1} = E$$

such that the following conditions hold: if (E, θ) is a semistable Higgs bundle, then this is the usual generalization of the HN filtration to Higgs bundles so that the

bundles $(F_i/F_{i-1}, \theta_i/\theta_{i-1})$ are stable Higgs bundles of slope $= \mu_M(E, \theta) = \mu(E)$. Otherwise

- (a) for $1 \leq i \leq n$, the quotients $(F_i/F_{i-1}, \theta_i/\theta_{i-1})$ are stable Higgs bundles each of slope $\mu_M(E, \theta)$,
- (b) ϕ projects to some non-zero $\psi \in H^0(E/F_n)$ and $(E/F_n, \theta/\theta_n, \psi)$ is a stable Higgs triple, and $\mu_M(E/F_n, \theta/\theta_n) < \mu_M(E, \theta) < \mu_m(E/F_n, \theta/\theta_n, \psi)$,
- (c) E/F_n has minimal rank such that (a) and (b) are satisfied.

When (E, θ) is unstable, the subbundle E/F_n is uniquely determined by the graded object

$$gr^+(\bar{\partial}_E, \theta) = (E/F_n, \theta/\theta_n) \oplus (F_1/F_0, \theta_1/\theta_0) \oplus \cdots \oplus (F_n/F_{n-1}, \theta_n/\theta_{n-1})$$

and is unique up to isomorphism of $(F_1/F_0, \theta_1/\theta_0) \oplus \cdots \oplus (F_n/F_{n-1}, \theta_n/\theta_{n-1})$.

Proof. If $(\bar{\partial}_E, \theta)$ is a semistable Higgs bundle, then $\mu_M(\bar{\partial}_E, \theta) = \mu(E)$ and we simply use the usual HN filtration of the Higgs bundle $(\bar{\partial}_E, \theta)$. If $(\bar{\partial}_E, \theta)$ is unstable, then let F be the unique maximal destabilizing subbundle of $E^{\bar{\partial}_E}$. Note that F is θ -invariant and $(F, \theta|_F)$ is a semistable Higgs bundle. In this case we let $0 = F_0 \subset \cdots \subset F_n = F$ be the usual HN filtration for F and thus obtain the filtration as stated in the proposition.

Since $\mu(F) = \mu_M(\bar{\partial}_E, \theta)$ by the choice of F , part (a) follows directly. Part (b) follows from a similar argument to the corresponding statement in [6]. The only difference is that we consider Higgs field invariant subbundles instead of any subbundles, and all extensions are extensions of Higgs bundles as opposed to simply vector bundles. Part (c) follows from the choice of F . \square

We use the proposition to define a grading on Higgs triples as follows.

Definition 6.5. For any stable Higgs triple $(\bar{\partial}_E, \theta, \phi)$ such that $(\bar{\partial}_E, \theta)$ is unstable, we define the μ_+ grading to be given by

$$gr^+(\bar{\partial}_E, \theta, \phi) = (gr^+(\bar{\partial}_E, \theta), \psi),$$

where $gr^+(\bar{\partial}_E, \theta)$ and ψ are as above. If $(\bar{\partial}_E, \theta)$ is semistable,

$$gr^+(\bar{\partial}_E, \theta, \phi) = (Gr(\bar{\partial}_E, \theta), 0),$$

where $Gr(\bar{\partial}_E, \theta)$ is the usual HN filtration for Higgs bundles.

The two gradings gr^- and gr^+ will help us stratify the space $\hat{\mathcal{B}}$. At this point the reader may notice that for any triple $(\bar{\partial}_E, \theta, \phi)$, both gradings naturally yield $(n+2)$ -tuples of integers $(d_\phi, r_\phi, r_1, \dots, r_n)$ which satisfy all the conditions of Lemma 5.5. The numbers are obtained from the gradings in the following manner: given a gr^- grading, let d_ϕ and r_ϕ be the degree and rank of E_ϕ respectively, where r_ϕ is the rank of the bundle F_i/F_{i-1} . Given a gr^+ grading, let d_ϕ and r_ϕ be the degree and rank of the bundle F_{n+1}/F_n , where r_ϕ is the same as in gr^- . We now proceed to the stratification.

Definition 6.6. Given any $(n+2)$ -tuple $(d_\phi, r_\phi, r_1, \dots, r_n) \in \mathcal{I}_\tau$, let

$$\mathcal{W}^\pm(d_\phi, r_\phi, r_1, \dots, r_n) = \left\{ (\bar{\partial}_E, \theta, \phi) \in \hat{\mathcal{B}} \mid (\bar{\partial}_E, \theta, \phi) \text{ is a stable Higgs triple, and } \right. \\ \left. gr^\pm(\bar{\partial}_E, \theta, \phi) \in \mathcal{Z}(d_\phi, r_\phi, r_1, \dots, r_n) \right\} \cup \mathcal{Z}(d_\phi, r_\phi, r_1, \dots, r_n).$$

and

$$\mathcal{W}_\tau^\pm = \bigcup_{(d_\phi, r_\phi, r_1, \dots, r_n) \in \mathcal{I}_\tau} \mathcal{W}^\pm(d_\phi, r_\phi, r_1, \dots, r_n)$$

The subspaces \mathcal{W}_τ^\pm form a stratification of $\hat{\mathcal{B}}$. A similar result is also true for \mathcal{W}^- . Before proving the next proposition, we introduce the following notation: let $\mathcal{U}o_\tau^\pm$ be the corresponding stratifying spaces of τ -semistable pairs as in [6], and $\hat{\mathcal{B}}o$ the master space of all such semistable pairs.

Proposition 6.7. *If $r > 2$, then for critical values $\tau \in \left(\frac{d}{r}, \frac{d}{r-1}\right)$, the complex codimension of \mathcal{W}_τ^\pm in $\hat{\mathcal{B}}$ is at least 1.*

Proof. We first consider \mathcal{W}_τ^+ . Let

$$\mathcal{W}o_\tau^+ = \left\{ (\bar{\partial}_E, \phi) \in \mathcal{A} \times H^0(E^{\bar{\partial}_E}) \mid (\bar{\partial}_E, \theta, \phi) \in \mathcal{W}_\tau^+ \right\}.$$

Then \mathcal{W}_τ^+ fibres over $\mathcal{W}o_\tau^+$ with fibre $\{(\bar{\partial}_E, \theta, \phi) \in \mathcal{W}_\tau^+ \mid \theta(\phi) = 0\}$ over the point $(\bar{\partial}_E, \phi) \in \mathcal{W}o_\tau^+$. Similarly, if we let

$$\mathcal{U}_\tau^+ = \left\{ (\bar{\partial}_E, \theta, \phi) \in \mathcal{W}_\tau^+ \mid (\bar{\partial}_E, \phi) \text{ is } \tau\text{-semistable as a pair} \right\},$$

then the space \mathcal{U}_τ^+ fibres over $\mathcal{U}o_\tau^+$. However, the space $\mathcal{U}o_\tau^+$ of τ -semi-stable pairs is an open subset of $\mathcal{W}o_\tau^+$. Moreover, the space $\mathcal{U}o_\tau^+$ has positive codimension in $\hat{\mathcal{B}}o$, the master space of pairs. Since $\hat{\mathcal{B}}$ fibres in a similar manner over $\hat{\mathcal{B}}o$, we see that the space \mathcal{U}_τ^+ must have positive codimension in $\hat{\mathcal{B}}$, and hence so must \mathcal{W}_τ^+ . We summarize the above argument in the following diagram.

$$\begin{array}{ccccc} \hat{\mathcal{B}} & \longleftarrow & \mathcal{U}_\tau^+ & \longrightarrow & \mathcal{W}_\tau^+ \\ \downarrow & & \downarrow & & \downarrow \\ \hat{\mathcal{B}}o & \xleftarrow{\text{pos.codim.}} & \mathcal{U}o_\tau^+ & \xrightarrow{\text{open}} & \mathcal{W}o_\tau^+ \end{array}$$

The same argument works for the case \mathcal{W}_τ^- . □

7. MORSE THEORY AND BIRATIONAL EQUIVALENCE

We now look more carefully at the Morse theory of the function f . We will see that the stratification obtained using Morse theory coincides with the algebraic stratification of the previous section. We first define the flow on $\hat{\mathcal{B}}$ as follows.

Definition 7.1. *Let $\Phi : \hat{\mathcal{B}} \times (-\infty, \infty) \rightarrow \hat{\mathcal{B}}$ be the flow*

$$\Phi_t([\bar{\partial}_E, \theta, \phi]) = [\bar{\partial}_E, \theta, e^{-t/2\pi r} \phi].$$

Note that since the flow only affects the section ϕ , our situation is identical to the one using stable pairs in [6].

Proposition 7.2. *Φ is continuous and it preserves $\hat{\mathcal{B}}_0$. Further, it coincides with the gradient flow of f on $\hat{\mathcal{B}}_0$ i.e.,*

$$\frac{d\Phi_t}{dt} = -\nabla_{\Phi_t} f.$$

Proof. This is similar to proposition 4.1 in [6]. Since the tangent space $T_{[\bar{\partial}_E, \theta, \phi] \hat{\mathcal{B}}_0}$ has been identified with $H^1(\mathbf{C}_0^{(\bar{\partial}_E, \theta, \phi)})$, the infinitesimal vector field of the S^1 action on $\hat{\mathcal{B}}_0$ is given by $\xi^\#[\bar{\partial}_E, \theta, \phi] = \frac{i}{r}(0, 0, r)$. Hence,

$$\begin{aligned} \nabla_{\Phi_t[\bar{\partial}_E, \theta, \phi]} f &= \frac{-1}{2\pi i} \nabla_{\Phi_t[\bar{\partial}_E, \theta, \phi]} \Psi = \frac{1}{2\pi i} \xi^\#(\Phi_t[\bar{\partial}_E, \theta, \phi]) \\ &= \frac{1}{2\pi r} (0, 0, e^{-t/2\pi r} \phi) = -\frac{d}{dt} (0, 0, e^{-t/2\pi r} \phi) \\ &= -\frac{d\Phi_t[\bar{\partial}_E, \theta, \phi]}{dt}. \end{aligned}$$

□

Definition 7.3. Given a critical value τ , and $(d_\phi, r_\phi, r_1, \dots, r_n) \in \mathcal{I}_\tau$, let

$$\mathcal{W}^s(d_\phi, r_\phi, r_1, \dots, r_n) = \left\{ [\bar{\partial}_E, \theta, \phi] \in \hat{\mathcal{B}} \mid \lim_{t \rightarrow \infty} \Phi_t([\bar{\partial}_E, \theta, \phi]) \in \mathcal{Z}(d_\phi, r_\phi, r_1, \dots, r_n) \right\}.$$

We similarly define $\mathcal{W}^u(d_\phi, r_\phi, r_1, \dots, r_n)$ by taking the limit as $t \rightarrow -\infty$. This allows us to define

$$\mathcal{W}_\tau^s = \bigcup_{(d_\phi, r_\phi, r_1, \dots, r_n) \in \mathcal{I}_\tau} \mathcal{W}^s(d_\phi, r_\phi, r_1, \dots, r_n)$$

and similarly,

$$\mathcal{W}_\tau^u = \bigcup_{(d_\phi, r_\phi, r_1, \dots, r_n) \in \mathcal{I}_\tau} \mathcal{W}^u(d_\phi, r_\phi, r_1, \dots, r_n).$$

We shall call \mathcal{W}^s and \mathcal{W}^u the stable and unstable Morse stratifications of $\hat{\mathcal{B}}$, respectively.

Proposition 7.4. For every critical value τ , the Morse stratification of $\hat{\mathcal{B}}$ coincides exactly with the algebraic stratification of $\hat{\mathcal{B}}$ i.e.,

$$\mathcal{W}_\tau^s = \mathcal{W}_\tau^+ \quad \text{and} \quad \mathcal{W}_\tau^u = \mathcal{W}_\tau^-.$$

Proof. This is similar to proposition 4.3 in [6]. First consider \mathcal{W}_τ^u and \mathcal{W}_τ^- . Since both stratify $\hat{\mathcal{B}}$, it is sufficient to show that $\mathcal{W}_\tau^- \subseteq \mathcal{W}_\tau^u$ for all τ . In particular, it is enough to show that

$$\mathcal{W}_\tau^-(d_\phi, r_\phi, r_1, \dots, r_n) \subseteq \mathcal{W}_\tau^u(d_\phi, r_\phi, r_1, \dots, r_n)$$

for all $(d_\phi, r_\phi, r_1, \dots, r_n) \in \mathcal{I}_\tau$. Consider any $[\bar{\partial}_E, \theta, \phi] \in \mathcal{W}_\tau^-(d_\phi, r_\phi, r_1, \dots, r_n)$. Let

$$0 = E_\phi = F_0 \subset F_1 \subset \dots \subset F_n = E$$

denote the μ_- filtration of $(\bar{\partial}_E, \theta, \phi)$. Fix real numbers $0 < \mu_1 < \mu_2 < \dots < \mu_n$ such that $\sum_{i=1}^n r_i \mu_i = r_\phi$, and consider the 1-parameter subgroup of gauge transformations in $\mathcal{G}_0^{\mathbb{C}}$,

$$g_t = \begin{pmatrix} e^{t/2\pi} & 0 & 0 & \dots & 0 \\ 0 & e^{-t\mu_1/2\pi r} & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & 0 & e^{-t\mu_n/2\pi r} \end{pmatrix}$$

written with respect to the filtration above. Then

$$\lim_{t \rightarrow -\infty} \Phi_t[\bar{\partial}_E, \theta, \phi] = \lim_{t \rightarrow -\infty} [\bar{\partial}_E, \theta, g_t^{-1}\phi] = \lim_{t \rightarrow -\infty} [g_t(\bar{\partial}_E, \theta), \phi] = [gr^-(\bar{\partial}_E, \theta), \phi].$$

The other case is similar. Again, it suffices to show that

$$\mathcal{W}_\tau^+(d_\phi, r_\phi, r_1, \dots, r_n) \subseteq \mathcal{W}_\tau^s(d_\phi, r_\phi, r_1, \dots, r_n)$$

for all $(d_\phi, r_\phi, r_1, \dots, r_n) \in \mathcal{I}_\tau$. So take any $[\bar{\partial}_E, \theta, \phi] \in \mathcal{W}_\tau^+(d_\phi, r_\phi, r_1, \dots, r_n)$. Let

$$0 = F_0 \subset F_1 \subset \dots \subset F_{n+1} = E$$

denote the μ_+ filtration of $(\bar{\partial}_E, \theta, \phi)$. If $(\bar{\partial}_E, \theta)$ is semistable (as a Higgs bundle), fix real numbers $1 > \mu_1 > \mu_2 > \dots > \mu_{n+1}$ such that $\sum_{i=1}^n r_i \mu_i = 0$, and let g_t be the gauge transformation

$$g_t = \begin{pmatrix} e^{t\mu_1/2\pi r} & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & e^{-t\mu_{n+1}/2\pi r} \end{pmatrix}$$

written with respect to the filtration above. If $(\bar{\partial}_E, \theta)$ is not semistable (as a Higgs bundle), then let $\mu_{n+1} = 1$, and choose the other μ_i such that $r_{n+1} + \sum_{i=1}^n r_i \mu_i = 0$. The rest of the argument proceeds as before. \square

We finally focus our attention on how the spaces \mathcal{B}_τ are affected as τ is varied smoothly. We start with a proposition which follows essentially from the Morse theory of f .

Theorem 7.5. (a) *If f has no critical values in the interval $[\tau, \tau + \varepsilon]$, then the Morse flow induces a biholomorphism between $\mathcal{B}_{\tau+\varepsilon}$ and \mathcal{B}_τ .*

(b) *If τ is the only critical value of f in the interval $[\tau, \tau + \varepsilon]$, then the Morse flow induces a biholomorphism between $\mathcal{B}_{\tau+\varepsilon} \setminus \mathbb{P}_\varepsilon(\mathcal{W}_\tau^+)$ and $\mathcal{B}_\tau \setminus \mathcal{Z}_\tau$, where*

$$\mathbb{P}_\varepsilon(\mathcal{W}_\tau^+) = (\mathcal{W}_\tau^+ \cap f^{-1}(\tau + \varepsilon))/U(1).$$

Proof. This proof follows the same argument as the corresponding statement, theorem 4.4 in [6]. As a result, we simply sketch here a construction of the biholomorphism.

Denote the equivalence class of the triple $[\bar{\partial}_E, \theta, \phi] \in \hat{\mathcal{B}}$ by x . Let

$$F : f^{-1}(\tau + \varepsilon) \times [0, \infty) \longrightarrow \mathbb{R}$$

be defined by $F(x, t) = f(\Phi_t(x))$. In part (a), since f is by assumption, smooth on the interval $[\tau, \tau + \varepsilon]$, we see that F is smooth. Further,

$$\left. \frac{\partial F}{\partial t} \right|_{(x,t)} = df_{\Phi_t(x)} \left(\left. \frac{\partial \Phi_t}{\partial x} \right|_x \right) = \|\nabla_{\Phi_t(x)} f\|^2 \neq 0,$$

for any $(x, t) \in F^{-1}(\tau)$ since $\tau = f(\Phi_t(x))$ is not a critical value of f . As a result, we may use the Implicit Function Theorem to solve the equation $F(x, t) = \tau$ and get $t = t(x)$ as a smooth function of x . This allows us to define a map:

$$(7.1) \quad \hat{\sigma}_+ : f^{-1}(\tau + \varepsilon) \longrightarrow f^{-1}(\tau)$$

by $\hat{\sigma}_+(x) = \Phi_{t(x)}(x)$. $\hat{\sigma}_+$ then induces a biholomorphism

$$\sigma : \mathcal{B}_{\tau+\varepsilon} = f^{-1}(\tau + \varepsilon)/U(1) \longrightarrow f^{-1}(\tau)/U(1) = \mathcal{B}_\tau.$$

The above follows once we establish that the map $\hat{\sigma}_+$ and the complex structure on the spaces $f^{-1}(\tau + \varepsilon)$, $f^{-1}(\tau)$ are all $U(1)$ -invariant. In part (b), the same argument as in part (a) gives a smooth map

$$\hat{\sigma}_+ : f^{-1}(\tau + \varepsilon) \setminus \mathcal{W}_\tau^+ \longrightarrow f^{-1}(\tau) \setminus \mathcal{Z}_\tau.$$

This map can then be extended (continuously) across \mathcal{W}_τ^+ by setting $\hat{\sigma}_+(x) = \lim_{t \rightarrow \infty} \Phi_t(x)$ for $x \in \mathcal{W}_\tau^+$. Finally, the same argument as before shows that $\hat{\sigma}_+$ is a biholomorphism onto its image away from $\mathbb{P}_\varepsilon(\mathcal{W}_\tau^+) = (\mathcal{W}_\tau^+ \cap f^{-1}(\tau + \varepsilon))/U(1)$. \square

Note that if the flow lines are reversed, the above argument shows that a similar statement must be true for $\mathcal{B}_{\tau-\varepsilon}$ and \mathcal{B}_τ . This observation leads us to the following.

Corollary 7.6. *If τ is the only critical value in $[\tau - \varepsilon, \tau + \varepsilon]$, then $\mathcal{B}_{\tau-\varepsilon}$ and $\mathcal{B}_{\tau+\varepsilon}$ are related as shown:*

$$\begin{array}{ccc} \mathcal{B}_{\tau-\varepsilon} & & \mathcal{B}_{\tau+\varepsilon} \\ & \searrow \sigma_- \quad \swarrow \sigma_+ & \\ & \mathcal{B}_\tau & \end{array}$$

where σ_\pm are continuous, and $\sigma_\pm : \mathcal{B}_{\tau \pm \varepsilon} \setminus \sigma_\pm^{-1}(\mathcal{Z}_\tau) \longrightarrow \mathcal{B}_\tau \setminus \mathcal{Z}_\tau$ are biholomorphisms.

Theorem 7.7. *For all noncritical values of τ in $\left(\frac{d}{r}, \frac{d}{r-1}\right)$, the spaces \mathcal{B}_τ are all birational.*

Proof. By the previous corollary, the complex manifolds $\mathcal{B}_{\tau \pm \varepsilon} \setminus \sigma_\pm^{-1}(\mathcal{Z}_\tau)$ are biholomorphic. Let $\tau \in \left(\frac{d}{r}, \frac{d}{r-1}\right)$ be a critical value. Let

$$\begin{aligned} T_- &= \left\{ (\bar{\partial}_E, \phi) \mid (\bar{\partial}_E, \theta, \phi) \in \mathcal{B}_{\tau-\varepsilon} \setminus \sigma_-^{-1}(\mathcal{Z}_\tau) \text{ for some Higgs field } \theta \right\} \\ T_+ &= \left\{ (\bar{\partial}_E, \phi) \mid (\bar{\partial}_E, \theta, \phi) \in \mathcal{B}_{\tau+\varepsilon} \setminus \sigma_+^{-1}(\mathcal{Z}_\tau) \text{ for some Higgs field } \theta \right\} \end{aligned}$$

Then $\mathcal{B}_{\tau-\varepsilon} \setminus \sigma_-^{-1}(\mathcal{Z}_\tau)$ is a fibration over T_- and $\mathcal{B}_{\tau+\varepsilon} \setminus \sigma_+^{-1}(\mathcal{Z}_\tau)$ is a fibration over T_+ . However, the sets of $(\tau \pm \varepsilon)$ -stable Bradlow pairs are open subsets of T_\pm . Since these spaces of Bradlow pairs are birational from [6], it follows that T_- is birational to T_+ . Finally, our biholomorphism is an extension by identity (as θ is untouched by the Morse flow) of the map $T_- \rightarrow T_+$ to the fibres of $\mathcal{B}_{\tau \pm \varepsilon} \setminus \sigma_\pm^{-1}(\mathcal{Z}_\tau)$ over T_\pm . It follows that $\mathcal{B}_{\tau \pm \varepsilon}$ are birational. This completes the proof. \square

Now we recall that $\mathcal{B}_{\frac{d}{r}}$ and $\mathcal{B}_{\frac{d}{r-1}}$ are moduli spaces of Higgs bundles of rank r and $r - 1$, respectively. Let

$$\mathcal{H}_0(r, d) = \left\{ (\bar{\partial}_E, \theta) \in \mathcal{H}(r, d) \mid \det(\theta) = 0 \right\}$$

We see that $\mathcal{B}_{\frac{d}{r}+\varepsilon}$ is a fibration over \mathcal{H}_0 . We shall denote the resulting fibre space by \mathcal{F} . At the other end we have a similar situation: we see that $\mathcal{B}_{\frac{d}{r-1}-\varepsilon}$ is a projective bundle over $\mathcal{M}_{Higgs}(r-1, d)$ with fibre over (E', θ) , the projectivization of Higgs extensions of (E', θ) by $(\mathcal{O}, \text{const.})$. Denote this bundle by \mathcal{E} . This proves our second result, and gives us a geometric relationship between the moduli spaces of Higgs bundles of rank r and $r - 1$.

Theorem 7.8. *The fibre space \mathcal{F} over $\mathcal{H}_0(r, d)$ is birational to the projective bundle \mathcal{E} over $\mathcal{H}(r - 1, d)$.*

Although we have successfully established a geometric correspondence between moduli spaces of Higgs bundles, some questions still remain unanswered. For instance, an explicit description of the space \mathcal{E} and \mathcal{F} would be useful. In general, the space \mathcal{F} appears to have rather badly behaved fibres. However, we conjecture that these fibres have constant dimension over the nilpotent cone, which sits inside \mathcal{H}_0 . Another motivation in obtaining this result is to understand the torus action on the moduli space of Higgs bundles using induction on rank. We hope to study this in the near future as well.

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Current address: Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, D-04103 Leipzig, Germany

E-mail address: `mehta@mis.mpg.de`